JOURNAL OF ALGEBRA 18, 426–431 (1971)

Determination of Locally Finite Groups with Duals

GIOVANNI ZACHER

Seminario Matematico, Università, 35100 Padova

Communicated by G. Zappa

Received June 6, 1970

1. A group G is said to have a *dual* if there exists a group $\bar{G}$ and a dual-isomorphism of the lattice of all subgroups of G onto the lattice of all subgroups of $\bar{G}$.

Baer has shown that a group with dual is periodic and has determined the structure of the abelian groups with duals [1]. Nilpotent groups and finite solvable groups with duals have been characterized by Suzuki [5]; in [7] is proved that finite groups with duals are solvable. In [6] the structure of the solvable groups with duals has been determined; it turns out that their lattice is autodual. In [8] is shown that a locally solvable group with dual is solvable.

Our object here is to prove that also *a locally finite group with dual is solvable.*

2. (Notation and basic results.) $L(G)$ lattice of all subgroups of G, $\{1\}$ identical subgroup, $\langle x \rangle$ cyclic group generated by x, $Z(p^\infty)$ quasi-cyclic p-group, $C_K(H)$ centralizer of H in K, $Z(H)$ center of H, H' derived group, $H \lhd K$ H normal in K; $\bar{x}$ the lattice automorphism of G induced by the inner automorphism defined by the element x.

We shall frequently use the following results.

(2.1). *If a finite or a locally solvable group G is dual-isomorphic to a group $\bar{G}$, then G and $\bar{G}$ are both solvable; a solvable group has a dual if and only if it is a direct product of finite groups of mutually prime orders, each direct factor being either a P-group or a non Hamiltonian modular p-group.*

(2.2). *Let H be a locally finite not locally solvable group. Then H contains a lattice invariant perfect subgroup G with H/G solvable; any automorphism of $L(G)$ preserves the orders of subgroups.*

(2.3). *Let N be a normal subgroup of a finite group G, and φ a projectivity of G. N^φ is a normal subgroup of G^φ if one of the following conditions is satisfied:*

1. *G/N is perfect.*
2. *N is perfect.*
3. *The order of N is relatively prime to the index $[G : N]$ and φ preserves the orders of subgroups.*
4. *G/N contains no proper solvable normal subgroup.*

If G is perfect, then G^φ is perfect and $(Z(G))^\varphi = Z(G^\varphi)$.

(2.4). *Let G be a locally finite group and H a finite perfect subgroup with $Z(H) = \{1\}$. Then for any automorphism ψ of $L(G)$, $C_G(H^\psi) = (C_G(H))^\psi$.*

(2.5). *The lattice $L(G)$ of subgroups of the group G is a direct product of the nontrivial lattices L_1 and L_2 if and only if G is a direct product of groups G_1 and G_2 such that $L(G_i) \simeq L_i$ and the order of any element of G_1 is finite and relatively prime to the order of any element of G_2.*

A proof of the first statement of (2.1) can be found in [7] and [8], of the second one in [6]; a proof of (2.2) is given in [8, p. 229], of (2.3) in [5, pp. 50–51], of (2.5) in [5, p. 5]; (2.4) is an easy consequence of (2.3).

3. Now we show:

LEMMA 1. *Let the locally finite group G be dual-isomorphic to the group $\bar{G}$. If G has no normal solvable subgroup $N \neq \{1\}$, then $\bar{G}$ is perfect.*

Proof. Let φ be a dual-isomorphism of G onto $\bar{G}$, and assume $\bar{G} \neq \bar{G}'$; take a subgroup H of G of order prime p such that $\bar{G}' \leqslant H^\varphi < \bar{G}$. Let x be an element of G such that $H^x \neq H$; then $H^{x\varphi} \wedge H^\varphi$ is a maximal subgroup of H^φ and $H^{x\varphi}$, hence H and H^x are maximal subgroups of the finite group $D_x = H \vee H^x$; it follows that D_x is either a group of order p^2 or else a group of order pq^α (p, q distinct primes) where the q-Sylow subgroup Q is a normal elementary abelian group. Let L be a subgroup of order q of D_x; since $Q^\varphi \vee (L^\varphi \wedge H^\varphi) = (Q^\varphi \vee H^\varphi) \wedge L^\varphi$ we conclude that $Q \wedge (L \vee H) = (Q \wedge H) \vee L = L$, hence $Q = L$ and D_x is a nonabelian group of order pq. But then there exists a subgroup K of D_x such that $H \vee K = H \vee H^{\bar{x}}$, $K \wedge H^{\bar{x}} = \{1\}$; therefore $D_x^\varphi = H^\varphi \wedge K^\varphi = H^\varphi \wedge H^{\bar{x}\varphi} \lhd H^{\bar{x}\varphi} \vee K^\varphi = \bar{G}$, and $\bar{G}/D_x^\varphi$ is a P-group. We conclude that the normal group $N = \bigvee_{x \in G} H^{\bar{x}} = \bigvee_{x\in G} D_x$ is solvable, since it is dual-isomorphic to the solvable group $\bar{G}/\bigwedge_{x\in G} D_x^\varphi$ (2.1); this contradiction proves the lemma.

LEMMA 2. *A locally finite group with dual and finite 2-Sylow subgroups is solvable.*

Proof. Let A be a locally finite not locally solvable group with dual and let $R(A)$ denote the subgroup generated by all normal locally solvable subgroups of A. $A/R(A)$ is by (2.1) not identical, has a dual and does not contain a proper normal solvable subgroup. Let us assume the lemma not true. We choose among the locally finite not locally solvable groups H with duals, in which $R(H) = \{1\}$ and the 2-Sylow subgroups are finite, a group G in which the order 2^α of the 2-Sylow subgroups is as small as possible. By the Feit–Thompson theorem [2] and by (2.1), we have $\alpha > 1$ and G of infinite order; by (2.2) we may assume G also perfect. Let us distinguish two cases:

(i) In G there is an element c of order 2 which permutes with a non-identical element of odd order.

Denote with U_c the group generated by all elements of prime order which permute with c; the group $U_c/\langle c\rangle$ has a dual by (2.2), hence is solvable. Therefore in $U_c/\langle c\rangle$ a 2-Sylow subgroup has a normal nonidentical complement $N/\langle c\rangle$ (2.2). It follows that $N = \langle c\rangle \times L$, with $L \neq \{1\}$ and with all its elements of odd order. If now ψ is an automorphism of $L(G)$, then $L^\psi = L$ as soon as $\langle c\rangle^\psi = \langle c\rangle$. We conclude that $N(L^\varphi) \geqslant \langle c\rangle^\varphi$ where φ denotes a dual-isomorphism of G onto $\bar{G}$, and since $\langle c\rangle^\varphi \vee L = \bar{G}$ we get $L^\varphi \lhd \bar{G}$. The group L is locally solvable [2] and has a dual, hence $\bar{G}/L$ is solvable (2.1); but $\bar{G}$ is perfect (Lemma 1), therefore $\bar{G} = L^\varphi$, that is $L = \{1\}$, a contradiction.

(ii) In G the centralizer of any element of order 2 is a 2-group.

Take an element c of order prime $p \neq 2$, and denote with U_c the group generated by all elements of G of order prime which permute with c. $U_c/\langle c\rangle$ is locally solvable [2] with dual, hence (2.1) its Sylow subgroups are of finite order. It follows that if G has a p-Sylow subgroup S of infinite order, S contains a $Z(p^\infty)$-group H. Let then $\langle a\rangle$ be the subgroup of order p of H, and set $U = \bigvee_{x\in\langle a\rangle^\varphi} (H)^{\psi_x}$ where $\psi_x = \varphi\bar{x}\varphi^{-1}$. $U/\langle a\rangle$ has dual and is locally solvable, hence its Sylow subgroups are finite, a contradiction to $H \leqslant U$. G is therefore a locally finite infinite group with finite Sylow subgroups. Since G contains an abelian group of infinite order [3], G contains elements of prime order, whose centralizers are not p-groups. Among these elements choose one c of minimal order, and denote as before with U_c the group generated by all elements of order prime which permute with c. $U_c/\langle c\rangle$ is locally solvable [2] with dual; using now the same argument as in (i), one again derives a contradiction. The lemma is thus completely proved.

LEMMA 3. *If G is a nonidentical locally finite group with dual, then $G' \neq G$.*

Proof. If $R(G)$, the subgroup generated by all locally solvable normal subgroups coincides with G, then G is a locally solvable group with dual, hence solvable (2.1) and therefore $G' \neq G$.

Assume now $R(G) \neq G$. Let φ be a dual-isomorphism of $\mathscr{L}(G)$ onto $\mathscr{L}(\bar{G})$; if we set $G_1 = G/R(G)$ and $\bar{G}_1 = (R(G))^{\varphi}$, then φ induces a dual-isomorphism φ_1 between the nonidentical group G_1 and the group $\bar{G}_1$, which is perfect (Lemma 1).

(a) If H is a nonlocally solvable subgroup of G, then $C = C_H(H)$ coincides with $Z(H)$ and is of finite order.

If H is not locally solvable, then H contains a finite perfect subgroup $B \neq \{1\}$, such that $B/R(B)$ is simple. Since $C_{G_1}(B) \geqslant C$, to conclude it is enough to prove (a) for finite perfect groups $H \neq \{1\}$, with $H/R(H)$ simple. We distinguish two cases:

(a_1) $R(H) = \{1\}$.

From $C \wedge H = \{1\}$, we get $C^{\varphi_1} \vee H^{\varphi_1} = \bar{G}_1$, and since for all automorphisms of $L(G)$, $(C(H))^{\psi} = C(H^{\psi})$ (2.4), we conclude that C^{φ_1} is normal in $\bar{G}_1$.

Set $L = \bigvee_{x \in C^{\varphi_1}} (H)^{\psi_x}$, ($\psi_x = \varphi_1 \bar{x} \varphi_1^{-1}$); then $C^{\varphi_1} \leqslant N_{\bar{G}_1}(L^{\varphi_1})$ and L is a perfect group (2.3) with $C \leqslant C_{G_1}(L)$. If we set $A = L/L \wedge C$, $B = C/L \wedge C$, $\bar{A} = L^{\varphi_1}/L^{\varphi_1} \wedge C^{\varphi_1}$, $\bar{B} = C^{\varphi_1}/L^{\varphi_1} \wedge C^{\varphi_1}$, φ_1 determines a dual-isomorphism χ of $D = A \vee B = A \times B$ onto $\bar{D} = \bar{A} \vee \bar{B} = \bar{A} \times \bar{B}$, where $\bar{A} = A^{\chi}$, $\bar{B} = B^{\chi}$.

Assume the existence of a group $\bar{S} \leqslant \bar{D}$ of order p^2, p a prime, such that $\bar{S} \wedge \bar{A} = \langle \bar{a} \rangle$, $\bar{S} \wedge \bar{B} = \langle \bar{b} \rangle$, $\bar{a}$, $\bar{b}$ elements of order p. Put $\bar{P} = \langle \bar{a}\bar{b} \rangle$; then $P \vee B = D$ since $\bar{P} \wedge \bar{B} = \{1\}$. Hence the canonical projection of P into A is onto, and therefore $P \wedge A \lhd A$; now $A/P \wedge A$ is dual-isomorphic to $\bar{P} \vee \bar{A}/\bar{A} \simeq \langle \bar{a}\bar{b} \rangle$, hence $A/P \wedge A$ is of prime order, thus A is not perfect, a contradiction. We conclude that $L(\bar{A} \times \bar{B}) = L(\bar{A}) \times L(\bar{B})$ (2.5); but then also $L(D) = L(A) \times L(B)$ and therefore the orders of the elements of A and B are relative prime (2.5). Since the nonsolvable group H is isomorphic to a subgroup of A, we conclude that B is a locally solvable group [2]; therefore such is also C since $L \wedge C \leqslant Z(C)$. But C has a dual, thus $\bar{G}_1/C^{\varphi_1}$ is solvable, hence $C^{\varphi_1} = \bar{G}_1$, $C = \{1\}$ (2.1).

(a_2) $R(H) \neq 1$.

Set $C = C_{G_1}(H)$ and $Z = C \wedge H = Z(H)$. For $x \in H^{\varphi_1}$ we then get as consequence of (2.3) and (2.4): $Z = Z^{\psi_x} = H \wedge C^{\psi_x} \lhd H \vee C^{\psi_x}$, $H \vee C^{\psi_x}/Z = H/Z \times C^{\psi_x}/Z$. Therefore if $T = \bigvee_{x \in H^{\varphi_1}} C^{\psi_x}$, T/Z centralizes H/Z, hence

$Z \leqslant T \wedge H \leqslant R(H)$. We show that $R(H) = Z$ and hence $T \wedge H = Z$. In fact set $M = R(H)$ and $U = \bigvee_{x \in M^{\varphi_1}} H^{\psi_x}$, where $\psi_x = \varphi_1 \bar{x} \varphi_1^{-1}$; then $M \lhd U$ by (2.3), and U/M has dual. The group U is perfect since generated by perfect groups (2.3). $C = C_U(M)$ has finite index in U, hence $U/M \vee C$ is a finite perfect group with dual, thus $U = M \vee C$. Since $U/C \simeq M/M \wedge C$, U/C is solvable, therefore $C = U$, thus $R(H) = Z(H)$. For $x \in Z^{\varphi_1}$ we then get $Z = Z^{\psi_x} = T^{\psi_x} \wedge H^{\psi_x} \lhd T^{\psi_x}$ since $H^{\psi_x} \lhd (T \vee H)^{\psi_x}$ (2.3); but also $Z = Z^{\psi_x} = Z(H^{\psi_x})$ (2.3), therefore $Z \lhd (T \vee H)^{\psi_x}$. We conclude that if we set $F = \bigvee_{x \in Z^{\varphi_1}} (HT)^{\psi_x}$, the group F/Z has dual; in F/Z, H/Z is simple not abelian, hence by (b_1), $C_{F/Z}(H/Z) = \{1\}$, and since $C_{F/Z}(H/Z) \geqslant T/Z \geqslant C/Z$, we conclude that $C = Z$.

(b) We are now able to derive a contradiction from the assumption $G_1' = G_1$.

G_1 is not solvable, hence its 2-Sylow subgroups are of infinite order (Lemma 2); let S denote such a group. S then contains an abelian group of infinite order [3], therefore S contains either an infinite elementary abelian group E or else S satisfies the minimal condition for subgroups (cf. [4], vol. II, p. 231), in which case S has a unique abelian divisible normal subgroup E of finite index. Write now, if possible, E as a direct product of two infinite groups E_1, E_2, and set $U_{E_i} = \bigvee_{x \in E_i^{\varphi_1}} E_j^{\psi_x}$, $i = 1, 2$, $i \neq j$. Since the automorphisms of $\mathscr{L}(G_1)$ are not singular (2.2), $C_{G_1}(U_{E_i}) \geqslant E_i$ and the group $L = E_1 \vee U_{E_1}/E_1$ has a dual and contains a group isomorphic to E_2; therefore L (2.1), and hence also U_{E_1} is not a locally solvable group; by (a) then $C_{G_1}(U_{E_i})$ is finite; but $C(U_{E_1}) \geqslant E_1$, hence a contradiction. We are left with the case in which each 2-Sylow subgroup S of G contains a unique normal subgroup E of finite index isomorphic to $Z(2^\infty)$. For given S, consider the normalizer $\mathscr{N}(H)$ in G of a subgroup H of E. It is not difficult to see, taking into account the structure of S, (2.2) and (2.3), that $(\mathscr{N}(H))^\psi = \mathscr{N}(H^\psi)$ for every automorphism of $\mathscr{L}(G)$. It follows that $\mathscr{N}(E)/E$ has a dual, and since its 2-Sylow subgroups are finite, we get (Lemma 2 and (2.1)) $\mathscr{N}(E)/E = S/E \vee C/E$, where $C \lhd \mathscr{N}(E)$ and $C = E \times T$; we conclude that $T^{\psi_x} = T$ for $x \in E^{\varphi_1}$, hence $T^{\varphi_1} \lhd \bar{G}_1$ and $\bar{G}_1/T^{\varphi_1}$ is solvable (2.1); therefore $T^{\varphi_1} = \bar{G}_1$, since $\bar{G}_1$ is perfect (Lemma 1); it follows that $\mathscr{N}(E) = S$. Let H be a finite subgroup of E of order $\geqslant 4$. The infinite group $\mathscr{N}(H)/H$ has a dual, hence it cannot be a 2-group (2.1). Let a be an element of order prime $p \neq 2$, which permutes with the elements of H, and denote with U_a the subgroup of G generated by all the elements of order 4 which permute with a. Then the group $U_a \vee \langle a \rangle/\langle a \rangle$ has a dual (2.2) and the 2-Sylow subgroups are finite since $\mathscr{N}(E)$ is a 2-group. Thus $U_a \vee \langle a \rangle/\langle a \rangle = L/\langle a \rangle \times M/\langle a \rangle$, where $L/\langle a \rangle$ denotes the 2-Sylow subgroup of $U_a \vee \langle a \rangle/\langle a \rangle$ (2.1); but $L = \langle a \rangle \times P$ and therefore P is a normal 2-Sylow subgroup.

But then for $x \in \langle a \rangle^{\varphi_1}$, $P^{\psi_x} = P$ since $U_a^{\psi_x} = U_a$. We conclude that $\mathcal{N}(P^{\varphi_1}) \geqslant P^{\varphi_1} \vee \langle a \rangle^{\varphi_1} = \bar{G}_1$ and since $\bar{G}_1/P^{\varphi_1}$ is solvable (2.1) we must have $P^{\varphi_1} = \bar{G}_1$ i.e. $H \leqslant P = \{1\}$, a contradiction.

This last contradiction concludes the proof of the Lemma 3.

THEOREM. *A locally finite group G has a dual if and only if G is a direct product of finite groups of mutually prime orders each direct factor being either a P-group or a non Hamiltonian modular p-group.*

Proof. Necessity. Assume $R(G) \neq G$; then $G/R(G)$ contains a perfect nonidentical subgroup with dual (2.2). But this is impossible by Lemma 3; therefore $R(G) = G$, and the conclusion now follows from (2.1). For the sufficiency see (2.1).

REFERENCES

1. R. BAER, Duality and commutativity of groups, *Duke Math. J.* 5 (1939), 824–838.
2. W. FEIT AND J. THOMPSON, Solvability of groups of odd order, *Pacif. J. Math.* 13 (1963), 775–1029.
3. M. I. KARGAPOLOV, On a problem of O. Yu. Šmidt, *Sibir. Mat. Ž.* 4 (1963), 232–235.
4. A. G. KUROSH, "The Theory of Groups," Chelsea Pub. Co., New York, N. Y., 1956.
5. M. SUZUKI, "Structure of a Group and the Structure of its Lattice of Subgroups", Erg. der Math., Springer-Verlag, Berlin, 1956.
6. G. ZACHER, I gruppi risolubili con duale, *Rend. Sem. Mat. Padova* 31 (1961), 104–113.
7. G. ZACHER, Caratterizzazione dei gruppi immagini omomorfe duali di un gruppo finito, *Rend. Sem. Mat. Padova* 31 (1961), 412–422.
8. G. ZACHER, Sui gruppi localmente finiti con duale, *Rend. Sem. Mat. Padova* 36 (1966), 223–242.